

# Jacobi's Inversion Problem for Genus Two Hyperelliptic Integral

Kazuyasu Shigemoto<sup>1</sup>

## Abstract

Hinted by the elliptic parameterization of the Ising model, the addition formula of the elliptic function forms to give the integrable  $SU(2)$  group relation in the previous paper. We then expect that the addition formula of the Abelian function with any genus will form to give some integrable Lie group structure. In this paper, we study Jacobi's inversion problem for hyperelliptic integral with genus two and we expect some  $SU(2)$  structure for the addition formula of the hyperelliptic theta function with genus two.

Keywords: Ising model , integrability condition , Jacobi's inversion relation  
addition formula, hyperelliptic function, theta function

## Contents

I	Introduction
II	Jacobi's inversion problem
III	Theta identity of Riemann's type with two variables
IV	Parameterization of the ratio of theta function with the symmetric function of $x_1$ and $x_2$
V	Differential equation by using the addition formula of the theta function
VI	Differential equation for $\tau_{12} = 0$ case
VII	Summary and discussion
	Appendix A: Property of theta function
	Appendix B: Various theta identity
	Appendix C: Parameterization of the constant
	Appendix D: Parameterization of the ratio of other theta function
	Appendix E: Addition formula of theta function and differential formula

---

<sup>1</sup>E-mail address: shigemot@tezukayama-u.ac.jp

# 1 Introduction

There are many two dimensional integrable statistical models [1]. Let's consider the general Yang-Baxter equation of the spin model in the form

$$U(x)V(x, y)U(y) = V(y)U(y, x)V(x). \quad (1.1)$$

This type of Yang-Baxter equation is called the integrability condition. The meaning of the integrability condition is that the model has the group structure. As the Yang-Baxter equation of this type says that the product of three group actions for two different paths gives the same group action, the group structure of the model is expected to be  $SU(2)$ .

Furthermore if the Yang-Baxter equation satisfies the difference property such as

$$U(x)V(x + y)U(y) = V(y)U(x + y)V(x), \quad (1.2)$$

we can understand this relation as some kind of addition formula, and we can solve the problem exactly by using only the addition formula.

First and famous integrable and exactly solvable spin model is the Ising model[2]. The structure of the Ising model is  $SU(2)$ [1, 2]. While the Boltzmann weight of the Ising model can be parameterized by the elliptic function[1]. Then we expect the correspondence between the  $SU(2)$  group structure and the elliptic function. In other words, we expect that the symmetry of the elliptic function is  $SU(2)$ , and the origin of the addition formulae of the elliptic function comes from the  $SU(2)$  group structure. This is just the same as the addition formula of the trigonometric function, which is the Abelian function with genus zero. The addition formula of the trigonometric function forms to give the  $U(1)$  group structure  $\exp(i(x + y)) = \exp(ix) \exp(iy)$  by parameterizing the circle with the trigonometric function through the Euler's relation  $\exp(ix) = \cos(x) + i \sin(x)$ .

For the elliptic function, which is the Abelian function with genus one, addition formula closed as the rational function in the following complicated form

$$\operatorname{sn}(x + y) = \frac{\operatorname{sn}(x)\operatorname{cn}(y)\operatorname{dn}(y) + \operatorname{sn}(y)\operatorname{cn}(x)\operatorname{dn}(x)}{1 - k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)}, \quad (1.3)$$

$$\operatorname{cn}(x + y) = \frac{\operatorname{cn}(x)\operatorname{cn}(y) - \operatorname{sn}(x)\operatorname{sn}(y)\operatorname{dn}(x)\operatorname{dn}(y)}{1 - k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)}, \quad (1.4)$$

$$\operatorname{dn}(x + y) = \frac{\operatorname{dn}(x)\operatorname{dn}(y) - k^2\operatorname{sn}(x)\operatorname{sn}(y)\operatorname{cn}(x)\operatorname{cn}(y)}{1 - k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)}. \quad (1.5)$$

We expect the more precise structure than the closure of the rational function. As  $x$  and  $y$

are continuous variables, the closure of the algebraic function will be expected to give the Lie group structure. Then we expect that the addition formula Eq.(1.3), Eq.(1.4) and Eq.(1.5) will be transformed into the Lie group structure.

In the previous papers, we have considered the surface of the sphere, which has  $SU(2)$  symmetry, and we have parameterized the spherical trigonometry relations with the elliptic function. Then the addition formula of the elliptic function forms to give the integrable  $SU(2)$  Lie group structure of the Ising model in the form[3, 4]

$$U(x)V(x+y)U(y) = V(y)U(x+y)V(x), \quad (1.6)$$

$$U(x) = \exp\{iam(x, k)J_z\}, V(x) = \exp\{iam(kx, 1/k)J_x\}, \quad (1.7)$$

If we take  $k \rightarrow 0$ , Eq.(1.6) reduces to the addition formula of  $U(1)$  in the form  $\exp(ixJ_z)\exp(iyJ_z) = \exp(i(x+y)J_z)$ .

For spin 1/2 case, the above relation gives

$$U_{1/2}(x)V_{1/2}(x+y)U_{1/2}(y) = V_{1/2}(y)U_{1/2}(x+y)V_{1/2}(x), \quad (1.8)$$

$$\begin{aligned} U_{1/2}(x) &= \left( \sqrt{\frac{1 + \text{cn}(x, k)}{2}} + i\sigma_z \sqrt{\frac{1 - \text{cn}(x, k)}{2}} \right), \\ V_{1/2}(x) &= \left( \sqrt{\frac{1 + \text{cn}(kx, 1/k)}{2}} + i\sigma_x \sqrt{\frac{1 - \text{cn}(kx, 1/k)}{2}} \right) \\ &= \left( \sqrt{\frac{1 + \text{dn}(x, k)}{2}} + i\sigma_x \sqrt{\frac{1 - \text{dn}(x, k)}{2}} \right), \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (1.9)$$

For spin 1 case, the above relation gives

$$U_1(x)V_1(x+y)U_1(y) = V_1(y)U_1(x+y)V_1(x), \quad (1.10)$$

$$\begin{aligned} U_1(x) &= \left( 1 - (1 - \text{cn}(x, k))J_z^2 + iJ_z \text{sn}(x, k) \right), \\ V_1(x) &= \left( 1 - (1 - \text{cn}(kx, 1/k))J_x^2 + iJ_x \text{sn}(kx, 1/k) \right) \\ &= \left( 1 - (1 - \text{dn}(x, k))J_x^2 + iJ_x k \text{sn}(x, k) \right), \\ J_z &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \end{aligned} \quad (1.11)$$

In this way, the addition formula of the elliptic function forms to give the integrable  $SU(2)$  group structure.

For the Abelian function with genus two, we have 15 hyperelliptic functions  $f_i(x, y)$  ( $i = 1, \dots, 15$ ) and the addition formula becomes[10]

$$f_i(x_1 + x_2, y_1 + y_2) = \frac{F(f_1(x_1, y_1), f_1(x_2, y_2) \cdots f_{15}(x_1, y_1), f_{15}(x_2, y_2))}{G(f_1(x_1, y_1), f_1(x_2, y_2) \cdots f_{15}(x_1, y_1), f_{15}(x_2, y_2))}, (i = 1, \dots, 15) \quad (1.12)$$

where  $F$  and  $G$  are polynomial of the argument. We expect that this hyperelliptic addition formula will form some integrable Lie group structure. To have the hint of the structure of the addition formula of the hyperelliptic theta function, we first revisit the Jacobi's inversion problem of the hyperelliptic integral with genus two. This Jacobi's inversion problem is solved independently by Göpel[5, 6] and Rosenhain[7, 8, 9]. Rosenhain's paper is more precise so that we follow according to Rosenhain's paper.

## 2 Jacobi's inversion problem

### 2.1 Abelian integral with genus one case

We take the 4th order polynomial of the form  $f_4(x) = (1 - x^2)(1 - k^2x^2)$  and consider the Abelian integral

$$du = \frac{dx}{\sqrt{f_4(x)}}. \quad (2.1)$$

The Abelian function in this case are  $x$ ,  $\sqrt{1 - x^2}$ ,  $\sqrt{1 - k^2x^2}$ . Abelian function  $x$  is given as the inverse function of  $u$  from Eq.(2.1), which is given as one of the Jacobi's elliptic function  $x = \text{sn}(u, k)$ . Furthermore, Abelian function  $x = \text{sn}(u, k)$ ,  $\sqrt{1 - x^2} = \text{cn}(u, k)$ ,  $\sqrt{1 - k^2x^2} = \text{dn}(u, k)$  are expressed as the ratio of the theta function. For  $x = \text{sn}(u, k)$  case, we have

$$x = \text{sn}(u, k) = \text{sn}(u; \tau) = -\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0; \tau) \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z; \tau) / \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0; \tau) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z; \tau), \quad (2.2)$$

where

$$K = \int_0^1 \frac{dx}{\sqrt{f_4(x)}}, \quad K' = \sqrt{-1} \int_{1/k}^1 \frac{dx}{\sqrt{f_4(x)}}, \quad \tau = \sqrt{-1} K' / K, \quad (2.3)$$

$$k = \vartheta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0; \tau) / \vartheta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0; \tau), \quad u = 2Kz, \quad (2.4)$$

which connects  $\tau$  with  $k$ , and connects  $u$  with  $z$ .

## 2.2 Abelian integral with genus two case

We take the 5th order polynomial of the form  $f_5(x) = x(1-x)(1-k_0^2x)(1-k_1^2x)(1-k_2^2x)$  and consider the Abelian integral

$$du = \frac{(P + Qx_1)dx_1}{\sqrt{f_5(x_1)}} + \frac{(P + Qx_2)dx_2}{\sqrt{f_5(x_2)}}, \quad (2.5)$$

$$dv = \frac{(R + Sx_1)dx_1}{\sqrt{f_5(x_1)}} + \frac{(R + Sx_2)dx_2}{\sqrt{f_5(x_2)}}. \quad (2.6)$$

The Jacobi's inversion problem for genus two case is given in the followings: the single-valued function is the symmetric combination of  $x_1$  and  $x_2$ , and such single valued function is expressed as the ratio of the hyperelliptic theta function with the above two variables  $u$  and  $v$ .

For example,  $x_1x_2$  and  $(1-x_1)(1-x_2)$ , which are some of the symmetric combination of  $x_1$  and  $x_2$ , are given by

$$x_1x_2 = c_1\vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) / \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v), \quad (2.7)$$

$$(1-x_1)(1-x_2) = c_2\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v), \quad (2.8)$$

Similarly,  $(1-k_0^2x_1)(1-k_0^2x_2)$ ,  $(1-k_1^2x_1)(1-k_1^2x_2)$ ,  $(1-k_2^2x_1)(1-k_2^2x_2)$  are expressed as the ratio of the theta function with two variable  $u$  and  $v$ .

We will show the above statement by first parameterize the symmetric combination of  $x_1$  and  $x_2$  by using the theta identity, and next we derive the differential equation by using the addition formula of the theta function, which gives the differential equation in the form of Eq.(2.5) and Eq.(2.6).

### 3 Riemann theta identity with two variables

The theta function with two variables is defined by

$$\begin{aligned} & \vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (u, v; \tau_1, \tau_2, \tau_{12}) \\ &= \sum_{m, n \in \mathbb{Z}} \exp \left\{ \pi i \left( \tau_1 \left( m + \frac{a}{2} \right)^2 + \tau_2 \left( n + \frac{c}{2} \right)^2 + 2\tau_{12} \left( m + \frac{a}{2} \right) \left( n + \frac{c}{2} \right) \right) \right. \\ & \quad \left. + 2\pi i \left( \left( m + \frac{a}{2} \right) \left( u + \frac{b}{2} \right) + \left( n + \frac{c}{2} \right) \left( v + \frac{d}{2} \right) \right) \right\} \end{aligned} \quad (3.1)$$

$$\begin{aligned} &= \sum_{m, n \in \mathbb{Z}} \exp \left\{ \frac{\pi i}{4} \left( \tau_1 (2m + a)^2 + \tau_2 (2n + c)^2 + 2\tau_{12} (2m + a)(2n + c) \right. \right. \\ & \quad \left. \left. + 2(2m + a)(2u + b) + 2(2n + c)(2v + d) \right) \right\}. \end{aligned} \quad (3.2)$$

Using the above, we define

$$M(u, v) = \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i), \quad (3.3)$$

$$M'(u, v) = \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i), \quad (3.4)$$

$$M''(u, v) = \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u_i, v_i), \quad (3.5)$$

$$M'''(u, v) = \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (u_i, v_i). \quad (3.6)$$

Next we take the following combination

$$\begin{aligned} & M(u, v) + M'(u, v) \\ &= \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i) \\ &= \sum_{m'_i, n'_i \in \mathbb{Z}'} \exp \left\{ \frac{\pi i}{4} \sum_{i=1}^4 \left( \tau_1 m_i'^2 + \tau_2 n_i'^2 + 2\tau_{12} m'_i n'_i + 4m'_i u_i + 4n'_i v_i \right) \right\}, \end{aligned} \quad (3.7)$$

where  $m'_i, n'_i \in Z'$  means that  $(m'_1, m'_2, m'_3, m'_4)$  all take even integer or all take odd integer, and  $(n_1, n'_2, n'_3, n'_4)$  all take even integer or all take odd integer.

We define the Riemann matrix  $A$

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad (3.8)$$

which is the orthogonal matrix with  $A^T = A^{-1}$ .

Using this Riemann matrix, we transform  $m'_i, n'_i, u_i, v_i$  in the form

$$\begin{pmatrix} \tilde{m}_1 \\ \tilde{m}_2 \\ \tilde{m}_3 \\ \tilde{m}_4 \end{pmatrix} = A \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \\ m'_4 \end{pmatrix} = \begin{pmatrix} (m'_1 + m'_2 + m'_3 + m'_4)/2 \\ (m'_1 + m'_2 - m'_3 - m'_4)/2 \\ (m'_1 - m'_2 + m'_3 - m'_4)/2 \\ (m'_1 - m'_2 - m'_3 + m'_4)/2 \end{pmatrix}, \quad (3.9)$$

$$\begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \\ \tilde{n}_4 \end{pmatrix} = A \begin{pmatrix} n'_1 \\ n'_2 \\ n'_3 \\ n'_4 \end{pmatrix} = \begin{pmatrix} (n'_1 + n'_2 + n'_3 + n'_4)/2 \\ (n'_1 + n'_2 - n'_3 - n'_4)/2 \\ (n'_1 - n'_2 + n'_3 - n'_4)/2 \\ (n'_1 - n'_2 - n'_3 + n'_4)/2 \end{pmatrix}, \quad (3.10)$$

$$\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} (u_1 + u_2 + u_3 + u_4)/2 \\ (u_1 + u_2 - u_3 - u_4)/2 \\ (u_1 - u_2 + u_3 - u_4)/2 \\ (u_1 - u_2 - u_3 + u_4)/2 \end{pmatrix}, \quad (3.11)$$

$$\begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \end{pmatrix} = A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} (v_1 + v_2 + v_3 + v_4)/2 \\ (v_1 + v_2 - v_3 - v_4)/2 \\ (v_1 - v_2 + v_3 - v_4)/2 \\ (v_1 - v_2 - v_3 + v_4)/2 \end{pmatrix}. \quad (3.12)$$

Then we have

$$\begin{aligned} & M(u, v) + M'(u, v) \\ &= \sum_{\tilde{m}_i, \tilde{n}_i \in Z'} \exp \left\{ \frac{\pi i}{4} \sum_{i=1}^4 (\tau_1 \tilde{m}_i^2 + \tau_2 \tilde{n}_i^2 + 2\tau_{12} \tilde{m}_i \tilde{n}_i + 4\tilde{m}_i \tilde{u}_i + 4\tilde{n}_i \tilde{v}_i) \right\} \\ &= M(\tilde{u}, \tilde{v}) + M'(\tilde{u}, \tilde{v}). \end{aligned} \quad (3.13)$$

The reason why the above relation is satisfied is the followings: if we consider  $\tilde{m}_1 = (m'_1 + m'_2 + m'_3 + m'_4)/2$ , then as  $\{m'_1, m'_2, m'_3, m'_4\}$  are all even integer or are all odd integer,  $\tilde{m}_1$  becomes integer. Furthermore, if we consider  $\tilde{m}_2 = (m'_1 + m'_2 - m'_3 - m'_4)/2$ , then  $\tilde{m}_2$  also becomes integer, and  $\tilde{m}_1 - \tilde{m}_2 = m'_3 + m'_4 = (\text{even integer})$ , so that we have  $\{\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, \tilde{m}_4\}$  are all even integer or are all odd integer.

We simply denote  $M = M(u, v)$ ,  $M' = M'(u, v)$ ,  $M'' = M''(u, v)$ ,  $M''' = M'''(u, v)$  and  $\tilde{M} = M(\tilde{u}, \tilde{v})$ ,  $\tilde{M}' = M'(\tilde{u}, \tilde{v})$ ,  $\tilde{M}'' = M''(\tilde{u}, \tilde{v})$ ,  $\tilde{M}''' = M'''(\tilde{u}, \tilde{v})$ .

Then we have the Riemann theta identity for two variables in the form

$$M + M' = \tilde{M} + \tilde{M}', \quad (3.14)$$

that is,

$$\begin{aligned} & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i) \\ = & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i). \end{aligned} \quad (3.15)$$

Another Riemann theta identity is given by replacing the variable in Eq.(3.15). By replacing  $u_1 \rightarrow u_1 + 1/2$ ,  $u_2 \rightarrow u_2 + 1/2$ ,  $u_3 \rightarrow u_3 + 1/2$ ,  $u_4 \rightarrow u_4 + 1/2$ , which gives  $\tilde{u}_1 \rightarrow \tilde{u}_1 + 1$ ,  $\tilde{u}_2 \rightarrow \tilde{u}_2$ ,  $\tilde{u}_3 \rightarrow \tilde{u}_3$ ,  $\tilde{u}_4 \rightarrow \tilde{u}_4$ , and also  $v_i$  are not changed.

This gives

$$M'' + M''' = \tilde{M} - \tilde{M}', \quad (3.16)$$

that is,

$$\begin{aligned} & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u_i, v_i) \\ = & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) - \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) - \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i), \end{aligned} \quad (3.17)$$

where we use  $\vartheta \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix} (\tilde{u}_i + 1, \tilde{v}_i) = -\vartheta \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i)$ .

Further, we replace  $u_1 \rightarrow u_1 + 1$ ,  $u_2 \rightarrow u_2$ ,  $u_3 \rightarrow u_3$ ,  $u_4 \rightarrow u_4$ , which gives  $\tilde{u}_1 \rightarrow \tilde{u}_1 + 1/2$ ,  $\tilde{u}_2 \rightarrow \tilde{u}_2 + 1/2$ ,  $\tilde{u}_3 \rightarrow \tilde{u}_3 + 1/2$ ,  $\tilde{u}_4 \rightarrow \tilde{u}_4 + 1/2$ , in Eq.(3.15) and also  $v_i$  are not changed.

This gives

$$M - M' = \tilde{M}'' + \tilde{M}''', \quad (3.18)$$



that is,

$$\begin{aligned} & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i) - \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) - \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i) \\ = & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i). \end{aligned} \quad (3.19)$$

Further, we replace  $u_1 \rightarrow u_1 + 1$ ,  $u_2 \rightarrow u_2$ ,  $u_3 \rightarrow u_3$ ,  $u_4 \rightarrow u_4$ , which gives  $\tilde{u}_1 \rightarrow \tilde{u}_1 + 1/2$ ,  $\tilde{u}_2 \rightarrow \tilde{u}_2 + 1/2$ ,  $\tilde{u}_3 \rightarrow \tilde{u}_3 + 1/2$ ,  $\tilde{u}_4 \rightarrow \tilde{u}_4 + 1/2$ , in Eq.(3.17) and also  $v_i$  are not changed. This gives

$$-M'' + M''' = -\tilde{M}'' + \tilde{M}''', \quad (3.20)$$

that is,

$$\begin{aligned} & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (u_i, v_i) - \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u_i, v_i) - \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u_i, v_i) \\ = & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) - \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) - \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i). \end{aligned} \quad (3.21)$$

In summary, we have

$$2M = \tilde{M} + \tilde{M}' + \tilde{M}'' + \tilde{M}''', \quad (3.22)$$

$$2M' = \tilde{M} + \tilde{M}' - \tilde{M}'' - \tilde{M}''', \quad (3.23)$$

$$2M'' = \tilde{M} - \tilde{M}' + \tilde{M}'' - \tilde{M}''', \quad (3.24)$$

$$2M''' = \tilde{M} - \tilde{M}' - \tilde{M}'' + \tilde{M}'''. \quad (3.25)$$

We inversely solve the above in the following form

$$2\tilde{M} = M + M' + M'' + M''', \quad (3.26)$$

$$2\tilde{M}' = M + M' - M'' - M''', \quad (3.27)$$

$$2\tilde{M}'' = M - M' + M'' - M''', \quad (3.28)$$

$$2\tilde{M}''' = M - M' - M'' + M'''. \quad (3.29)$$

We can express the above relation in the form

$$\begin{pmatrix} M \\ M' \\ M'' \\ M''' \end{pmatrix} = A \begin{pmatrix} \tilde{M} \\ \tilde{M}' \\ \tilde{M}'' \\ \tilde{M}''' \end{pmatrix}, \quad \begin{pmatrix} \tilde{M} \\ \tilde{M}' \\ \tilde{M}'' \\ \tilde{M}''' \end{pmatrix} = A \begin{pmatrix} M \\ M' \\ M'' \\ M''' \end{pmatrix}. \quad (3.30)$$

## 4 Parameterization of the ratio of theta function with the symmetric function of $x_1$ and $x_2$

We parameterize the ratio of theta function with the symmetric function of  $x_1$  and  $x_2$ . The basis of the symmetric function of two variable  $x_1$  and  $x_2$  are  $\{x_1x_2, x_1 + x_2\}$ , so that the linear combination of three polynomial is dependent.

We take  $x_1x_2$ ,  $(1 - x_1)(1 - x_2)$ ,  $(1 - k_0^2x_1)(1 - k_0^2x_2)$ , we have the following identity

$$1 = k_0^2x_1x_2 - \frac{k_0^2}{1 - k_0^2}(1 - x_1)(1 - x_2) + \frac{1}{1 - k_0^2}(1 - k_0^2x_1)(1 - k_0^2x_2). \quad (4.1)$$

While we have the following theta identity

$$\begin{aligned} 1 &= \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) / \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \\ &+ \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \\ &- \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v), \end{aligned} \quad (4.2)$$

which is given by Eq.(B.3) by replacing  $u \rightarrow u + \tau/2 + 1/2$ ,  $v \rightarrow v + \tau_{12}/2 + 1/2$ . Then we have the parameterization

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = k_0^2x_1x_2, \quad (4.3)$$

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = -\frac{k_0^2}{1-k_0^2}(1-x_1)(1-x_2), \quad (4.4)$$

$$\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = -\frac{1}{1-k_0^2}(1-k_0^2 x_1)(1-k_0^2 x_2). \quad (4.5)$$

Similarly we use the identity

$$1 = k_1^2 x_1 x_2 - \frac{k_1^2}{1-k_1^2}(1-x_1)(1-x_2) + \frac{1}{1-k_1^2}(1-k_1^2 x_1)(1-k_1^2 x_2), \quad (4.6)$$

and

$$\begin{aligned} 1 &= \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) / \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \\ &+ \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v) / \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \\ &+ \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v) / \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v), \end{aligned} \quad (4.7)$$

which is derived from Eq.(B.1) by replacing  $u \rightarrow u + \tau/2 + 1/2$ ,  $v \rightarrow v + \tau_{12}/2 + 1/2$ . Then we have the parameterization

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = k_1^2 x_1 x_2, \quad (4.8)$$

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = -\frac{k_1^2}{1-k_1^2}(1-x_1)(1-x_2), \quad (4.9)$$

$$\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = \frac{1}{1-k_1^2} (1-k_1^2 x_1)(1-k_1^2 x_2). \quad (4.10)$$

Similarly we use the identity

$$1 = k_2^2 x_1 x_2 - \frac{k_2^2}{1-k_2^2} (1-x_1)(1-x_2) + \frac{1}{1-k_2^2} (1-k_2^2 x_1)(1-k_1^2 x_2), \quad (4.11)$$

and

$$\begin{aligned} 1 &= \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) / \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \\ &+ \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v) / \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \\ &+ \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) / \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v), \end{aligned} \quad (4.12)$$

which is derived from Eq.(B.2) by replacing  $u \rightarrow u + \tau/2 + 1/2$ ,  $v \rightarrow v + \tau_{12}/2 + 1/2$ . Then we have the parameterization

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = k_2^2 x_1 x_2, \quad (4.13)$$

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = -\frac{k_2^2}{1-k_2^2} (1-x_1)(1-x_2), \quad (4.14)$$

$$\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = \frac{1}{1-k_2^2} (1-k_2^2 x_1)(1-k_2^2 x_2). \quad (4.15)$$

From this parameterization, the theta function with zero argument and various  $\{k_0, k_1, \dots\}$

is connected. Some consistency conditions must be checked. The detail of these connections and the consistency of the above relations is explained in the Appendix C. The summary of Appendix C is given as follows

$$k_0^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (4.16)$$

$$k_1^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (4.17)$$

$$k_2^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (4.18)$$

$$k_0'^2 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (4.19)$$

$$k_1'^2 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (4.20)$$

$$k_2'^2 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (4.21)$$

$$k_{01}^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}, \quad (4.22)$$

$$k_{02}^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}, \quad (4.23)$$

$$k_{12}^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}. \quad (4.24)$$

Inversely, we express the ratio of theta function with zero argument by  $\{k_0, k_1, \dots\}$  in the form

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k_0 k'_2 k_{12}}{k_1 k_{02}}, \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k'_0 k_2 k_{01}}{k_1 k_{02}}, \quad (4.25)$$

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k'_0 k'_2}{k'_1}, \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k_0 k'_2 k_{01}}{k'_1 k_{02}}, \quad (4.26)$$

$$\frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k'_0 k_2 k_{12}}{k'_1 k_{02}}, \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k_0 k_2}{k_1}, \quad (4.27)$$

$$\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k_{01} k_{12}}{k_1 k'_1}, \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k_0 k'_0 k_{12}}{k_1 k'_1 k_{02}}, \quad (4.28)$$

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} = \frac{k_2 k'_2 k_{01}}{k_1 k'_1 k_{02}}. \quad (4.29)$$

Using these expressions, we have the parameterization of ratio of the theta function with the symmetric function of  $x_1$  and  $x_2$  in the form

$$1) \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = k_0 k_1 k_2 x_1 x_2, \quad (4.30)$$

$$2) \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{k_0 k_1 k_2}{k'_0 k'_1 k'_2} (1 - x_1)(1 - x_2), \quad (4.31)$$

$$3) \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{k_1 k_2}{k'_0 k_{01} k_{02}} (1 - k_0^2 x_1)(1 - k_0^2 x_2), \quad (4.32)$$

$$4) \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = \frac{k_0 k_2}{k'_1 k_{01} k_{12}} (1 - k_1^2 x_1)(1 - k_1^2 x_2), \quad (4.33)$$

$$5) \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = \frac{k_0 k_1}{k'_2 k_{02} k_{12}} (1 - k_2^2 x_1)(1 - k_2^2 x_2). \quad (4.34)$$

The parameterization of other ratio of theta function is given in Appendix D, and we have

$$6) \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{F_{01}(x_1)F_{01}(x_2)}{k'_0 k'_1 k'_2 (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{01}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{01}(x_2)} \right\}^2, \quad (4.35)$$

$$7) \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = \frac{k_1 k_2 F_{34}(x_1) F_{34}(x_2)}{k'_1 k'_2 k_{01} k_{02} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{34}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{34}(x_2)} \right\}^2, \quad (4.36)$$

$$8) \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{k_0 k_2 F_{24}(x_1) F_{24}(x_2)}{k'_0 k'_2 k_{01} k_{12} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{24}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{24}(x_2)} \right\}^2, \quad (4.37)$$

$$9) \quad \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{k_0 k_1 F_{23}(x_1) F_{23}(x_2)}{k'_0 k'_1 k_{02} k_{12} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{23}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{23}(x_2)} \right\}^2, \quad (4.38)$$

$$10) \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = \frac{k_0 F_{12}(x_1) F_{12}(x_2)}{k'_1 k'_2 k_{01} k_{02} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{12}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{12}(x_2)} \right\}^2, \quad (4.39)$$

$$11) \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{k_1 F_{13}(x_1) F_{13}(x_2)}{k'_0 k'_2 k_{01} k_{12} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{13}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{13}(x_2)} \right\}^2, \quad (4.40)$$

$$12) \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{k_2 F_{14}(x_1) F_{14}(x_2)}{k'_0 k'_1 k_{02} k_{12} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{14}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{14}(x_2)} \right\}^2, \quad (4.41)$$

$$13) \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{k_0 F_{02}(x_1) F_{02}(x_2)}{k'_0 k_{01} k_{02} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{02}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{02}(x_2)} \right\}^2, \quad (4.42)$$

$$14) \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = \frac{k_1 F_{03}(x_1) F_{03}(x_2)}{k'_1 k_{01} k_{12} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{03}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{03}(x_2)} \right\}^2, \quad (4.43)$$



$$15) \quad \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = \frac{k_2 F_{04}(x_1) F_{04}(x_2)}{k_2' k_{02} k_{12} (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{F_{04}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{04}(x_2)} \right\}^2, \quad (4.44)$$

where we use the following functions,  $F_{01}(x) = x(1 - x)$ ,  $F_{02}(x) = x(1 - k_0^2 x)$ ,  $F_{03}(x) = x(1 - k_1^2 x)$ ,  $F_{04}(x) = x(1 - k_2^2 x)$ ,  $F_{12}(x) = (1 - x)(1 - k_0^2 x)$ ,  $F_{13}(x) = (1 - x)(1 - k_1^2 x)$ ,  $F_{14}(x) = (1 - x)(1 - k_2^2 x)$ ,  $F_{23}(x) = (1 - k_0^2 x)(1 - k_1^2 x)$ ,  $F_{24}(x) = (1 - k_0^2 x)(1 - k_2^2 x)$ ,  $F_{34}(x) = (1 - k_1^2 x)(1 - k_2^2 x)$ .

## 5 Differential equation by using the addition formula of the theta function

As we parameterize the ratio of all theta functions with the symmetric function of  $x_1$  and  $x_2$ , we can connect  $dx_1$  and  $dx_2$  with  $du$  and  $dv$ . Here we will show that we have the following differential equation

$$du = \frac{(P + Qx_1) dx_1}{\sqrt{f_5(x_1)}} + \frac{(P + Qx_2) dx_2}{\sqrt{f_5(x_2)}}, \quad (5.1)$$

$$dv = \frac{(R + Sx_1) dx_1}{\sqrt{f_5(x_1)}} + \frac{(R + Sx_2) dx_2}{\sqrt{f_5(x_2)}}. \quad (5.2)$$

To show the above differential equation, we start from

$$\vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) = -\sqrt{k_0 k_1 k_2} \sqrt{x_1 x_2}, \quad (5.3)$$

$$\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) = \sqrt{-1} \sqrt{\frac{k_0 k_1 k_2}{k_0' k_1' k_2'}} \sqrt{(1 - x_1)(1 - x_2)}, \quad (5.4)$$

and calculate  $\partial_u \sqrt{x_1 x_2}$ ,  $\partial_u \sqrt{(1 - x_1)(1 - x_2)}$ ,  $\partial_v \sqrt{x_1 x_2}$ ,  $\partial_v \sqrt{(1 - x_1)(1 - x_2)}$ , that is, we calculate

$$\frac{\partial}{\partial u} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right), \quad \frac{\partial}{\partial u} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right), \quad (5.5)$$

$$\frac{\partial}{\partial v} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right), \frac{\partial}{\partial v} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right), \quad (5.6)$$

by using the addition theorem of theta function.

## 5.1 Differential equation

Using Eq.(E.6) in Appendix E, we have

$$\begin{aligned} \frac{\partial}{\partial u} \sqrt{x_1 x_2} &= \frac{(-1)}{\sqrt{k_0 k_1 k_2}} \frac{\partial}{\partial u} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right) \\ &= \frac{(-1)}{\sqrt{k_0 k_1 k_2}} \left\{ \frac{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0) \partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, 0)|_0}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0)} \frac{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u, v)}{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} \frac{\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v)}{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} \right. \\ &\quad \left. - \frac{\vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0, 0) \partial_u \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, 0)|_0}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0)} \frac{\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u, v)}{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} \frac{\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u, v)}{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} \right\}. \\ &= -\frac{a}{2} \left( \sqrt{\frac{x_2}{x_1}} \frac{(1 - k_2^2 x_2) \sqrt{f_5(x_1)}}{(x_2 - x_1)} - \sqrt{\frac{x_1}{x_2}} \frac{(1 - k_2^2 x_1) \sqrt{f_5(x_2)}}{(x_2 - x_1)} \right) \\ &\quad + \frac{b}{2} \left( \sqrt{\frac{x_2}{x_1}} \frac{(1 - k_1^2 x_2) \sqrt{f_5(x_1)}}{(x_2 - x_1)} - \sqrt{\frac{x_1}{x_2}} \frac{(1 - k_1^2 x_1) \sqrt{f_5(x_2)}}{(x_2 - x_1)} \right) \\ &= \frac{1}{2} \left( \sqrt{\frac{x_2}{x_1}} \frac{(A + Bx_2) \sqrt{f_5(x_1)}}{(x_2 - x_1)} - \sqrt{\frac{x_1}{x_2}} \frac{(A + Bx_1) \sqrt{f_5(x_2)}}{(x_2 - x_1)} \right), \end{aligned} \quad (5.7)$$

where we use Eq.(4.34), Eq.(4.44) and Eq.(4.33), Eq.(4.43). The constants  $a$ ,  $b$ ,  $A$ ,  $B$  are given by

$$a = \frac{2}{k_2' k_0 k_{12}} \frac{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0) \partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, 0)|_0}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0)}, \quad (5.8)$$

$$b = \frac{2}{k'_1 k_{01} k_{12}} \frac{\vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \partial_u \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,0)|_0}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}, \quad (5.9)$$

and  $A = -a + b$ ,  $B = (ak_2^2 - bk_1^2)$ .  
Similarly, using Eq.(E.10), we have

$$\begin{aligned} \frac{\partial}{\partial u} \sqrt{(1-x_1)(1-x_2)} &= \sqrt{-1} \sqrt{\frac{k'_0 k'_1 k'_2}{k_0 k_1 k_2}} \frac{\partial}{\partial u} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \right) \\ &= \sqrt{-1} \sqrt{\frac{k'_0 k'_1 k'_2}{k_0 k_1 k_2}} \left\{ - \frac{\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,0)|_0 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,v)}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \right. \\ &\quad \left. + \frac{\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \partial_u \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,0)|_0 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,v)}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \right\}. \\ &= \frac{\tilde{a}}{2} \left( \sqrt{\frac{1-x_2}{1-x_1}} \frac{(1-k_2^2 x_2) \sqrt{f_5(x_1)}}{(x_2-x_1)} - \sqrt{\frac{1-x_1}{1-x_2}} \frac{(1-k_2^2 x_1) \sqrt{f_5(x_2)}}{(x_2-x_1)} \right) \\ &\quad - \frac{\tilde{b}}{2} \left( \sqrt{\frac{1-x_2}{1-x_1}} \frac{(1-k_1^2 x_2) \sqrt{f_5(x_1)}}{(x_2-x_1)} - \sqrt{\frac{x_1}{x_2}} \frac{(1-k_1^2 x_1) \sqrt{f_5(x_2)}}{(x_2-x_1)} \right) \\ &= -\frac{1}{2} \left( \sqrt{\frac{1-x_2}{1-x_1}} \frac{(\tilde{A} + \tilde{B}x_2) \sqrt{f_5(x_1)}}{(x_2-x_1)} - \sqrt{\frac{1-x_1}{1-x_2}} \frac{(\tilde{A} + \tilde{B}x_1) \sqrt{f_5(x_2)}}{(x_2-x_1)} \right), \quad (5.10) \end{aligned}$$

where we use Eq.(4.34), Eq.(4.41) and Eq.(4.33), Eq.(4.40). The constants  $\tilde{a}$ ,  $\tilde{b}$ ,  $\tilde{A}$ ,  $\tilde{B}$ , are given by

$$\tilde{a} = \frac{2}{k_{02} k_{12}} \frac{\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,0)|_0}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0)}, \quad (5.11)$$

$$\tilde{b} = \frac{2}{k_{01}k_{12}} \frac{\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \partial_u \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,0)|_0}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0)}. \quad (5.12)$$

and  $\tilde{A} = -\tilde{a} + \tilde{b}$  and  $\tilde{B} = (\tilde{a}k_2^2 - \tilde{b}k_1^2)$ .

We can check that  $a = \tilde{a}$  by using Eq.(4.21) and we can check  $b = \tilde{b}$  by using Eq.(4.20).

Then we have  $\tilde{A} = A$  and  $\tilde{B} = B$ .

Then we have

$$\begin{aligned} \frac{\partial}{\partial u} \sqrt{x_1 x_2} &= \frac{1}{2} \sqrt{\frac{x_2}{x_1}} \frac{\partial x_1}{\partial u} + \frac{1}{2} \sqrt{\frac{x_1}{x_2}} \frac{\partial x_2}{\partial u} \\ &= \frac{1}{2} \left( \sqrt{\frac{x_2}{x_1}} \frac{(A + Bx_2) \sqrt{f_5(x_1)}}{(x_2 - x_1)} - \sqrt{\frac{x_1}{x_2}} \frac{(A + Bx_1) \sqrt{f_5(x_2)}}{(x_2 - x_1)} \right), \end{aligned} \quad (5.13)$$

$$\begin{aligned} \frac{\partial}{\partial u} \sqrt{(1 - x_1)(1 - x_2)} &= -\frac{1}{2} \sqrt{\frac{1 - x_2}{1 - x_1}} \frac{\partial x_1}{\partial u} - \frac{1}{2} \sqrt{\frac{1 - x_1}{1 - x_2}} \frac{\partial x_2}{\partial u} \\ &= -\frac{1}{2} \left( \sqrt{\frac{1 - x_2}{1 - x_1}} \frac{(A + Bx_2) \sqrt{f_5(x_1)}}{(x_2 - x_1)} - \sqrt{\frac{1 - x_1}{1 - x_2}} \frac{(A + Bx_1) \sqrt{f_5(x_2)}}{(x_2 - x_1)} \right), \end{aligned} \quad (5.14)$$

which gives

$$\frac{\partial x_1}{\partial u} = \frac{(A + Bx_2) \sqrt{f_5(x_1)}}{(x_2 - x_1)}, \quad \frac{\partial x_2}{\partial u} = -\frac{(A + Bx_1) \sqrt{f_5(x_2)}}{(x_2 - x_1)}. \quad (5.15)$$

Similarly we have

$$\frac{\partial x_1}{\partial v} = \frac{(C + Dx_2) \sqrt{f_5(x_1)}}{(x_2 - x_1)}, \quad \frac{\partial x_2}{\partial v} = -\frac{(C + Dx_1) \sqrt{f_5(x_2)}}{(x_2 - x_1)} \quad (5.16)$$

where

$$c = \frac{2}{k_2' k_{02} k_{12}} \frac{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \partial_v \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (0,v)|_0}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}, \quad (5.17)$$

$$d = \frac{2}{k'_1 k_{01} k_{12}} \frac{\vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0, 0) \partial_v \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (0, v)|_0}{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0)} \quad (5.18)$$

and  $C = -c + d$  and  $D = (ck_2^2 - dk_1^2)$

This gives the differential equation

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} \quad (5.19)$$

where

$$\alpha = \frac{(A + Bx_2)\sqrt{f_5(x_1)}}{(x_2 - x_1)}, \quad \beta = \frac{(C + Dx_2)\sqrt{f_5(x_1)}}{(x_2 - x_1)} \quad (5.20)$$

$$\gamma = -\frac{(A + Bx_1)\sqrt{f_5(x_2)}}{(x_2 - x_1)}, \quad \delta = -\frac{(C + Dx_1)\sqrt{f_5(x_2)}}{(x_2 - x_1)} \quad (5.21)$$

Then we have

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \quad (5.22)$$

where we have  $\alpha\delta - \beta\gamma = (AD - BC)\sqrt{f_5(x_1)}\sqrt{f_5(x_2)}/(x_2 - x_1)$ .

Then we finally find

$$du = -\frac{1}{(AD - BC)} \left( \frac{(C + Dx_1)}{\sqrt{f_5(x_1)}} dx_1 + \frac{(C + Dx_2)}{\sqrt{f_5(x_2)}} dx_2 \right) \quad (5.23)$$

$$dv = \frac{1}{(AD - BC)} \left( \frac{(A + Bx_1)}{\sqrt{f_5(x_1)}} dx_1 + \frac{(A + Bx_2)}{\sqrt{f_5(x_2)}} dx_2 \right) \quad (5.24)$$

Then we can write in the form

$$du = \frac{(P + Qx_1)dx_1}{\sqrt{f_5(x_1)}} + \frac{(P + Qx_2)dx_2}{\sqrt{f_5(x_2)}} \quad (5.25)$$

$$dv = \frac{(R + Sx_1)dx_1}{\sqrt{f_5(x_1)}} + \frac{(R + Sx_2)dx_2}{\sqrt{f_5(x_2)}} \quad (5.26)$$

where we denote  $P = -C/(AD - BC)$ ,  $Q = -D/(AD - BC)$ ,  $R = A/(AD - BC)$ ,  $S = B/(AD - BC)$ .

In this way, using the expression of the symmetric combination of  $x_1$  and  $x_2$  as the ratio of the hyperelliptic theta function with two variable, we derive the above differential equation Eq.(5.25) and Eq.(5.25). Then we have solved the Jacobi's inversion problem for genus two case, that is, *i*) the single-valued function is the symmetric combination of  $x_1$  and  $x_2$ , *ii*) symmetric combination of  $x_1$  and  $x_2$  is expressed as the ratio of the hyperelliptic theta function with two variables  $u$  and  $v$ .

## 6 Differential equation for $\tau_{12} = 0$ case

Here we consider the case  $\tau_{12} = 0$ , which gives

$$\vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (u, v) = \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (u) \vartheta \begin{bmatrix} c \\ d \end{bmatrix} (v). \quad (6.1)$$

Using the expression

$$\text{sn}(u, k) = -\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0) \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z) / \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z), \quad (6.2)$$

$$\text{cn}(u, k) = \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z) / \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z), \quad (6.3)$$

$$\text{dn}(u, k) = \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z) / \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z), \quad (6.4)$$

$$k = \vartheta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0) / \vartheta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0), \quad k' = \vartheta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) / \vartheta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0), \quad (6.5)$$

From the relation

$$\text{cn}^2(u, k) + \text{sn}^2(u, k) = 1, \quad \text{dn}^2(u, k) + k^2 \text{sn}^2(u, k) = 1, \quad (6.6)$$

we have three identities

$$\vartheta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z) = \vartheta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z) + \vartheta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z), \quad (6.7)$$

$$\vartheta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z) = \vartheta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z) - \vartheta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z), \quad (6.8)$$

$$\vartheta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z) = \vartheta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z) + \vartheta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0) \vartheta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z), \quad (6.9)$$

with

$$\vartheta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0) = \vartheta^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) + \vartheta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0). \quad (6.10)$$

The fundamental theta identity of Eq.(B.1), Eq.(B.2) and Eq.(B.3) reduced to Eq.(6.7).

We define

$$x(= -\text{sn}(u, k)) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0) \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (u) / \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u), \quad (6.11)$$

$$y(= -\text{sn}(v, k)) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0) \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (v) / \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (v), \quad (6.12)$$

and we express  $x_1$  and  $x_2$  in Eq.(4.3), Eq.(4.4) and Eq.(4.5) with  $x$  and  $y$ . Straightforward calculation, we have

$$k_0^2 x_1 x_2 = x^2, \quad \frac{k_0^2}{1 - k_0^2} (1 - x_1)(1 - x_2) = -(1 - x^2), \quad \frac{(1 - k_0^2 x_1)(1 - k_0^2 x_2)}{1 - k_0^2} = 0. \quad (6.13)$$

Then we have two solutions

$$i) x_1 = x^2, \quad x_2 = \frac{1}{k_0^2} \quad ii) x_1 = \frac{1}{k_0^2}, \quad x_2 = x^2, \quad (6.14)$$

by cancel out the  $v$ -dependence. Similarly, from Eq.(4.8), Eq.(4.9) and Eq.(4.10), we have two solutions

$$i) x_1 = x^2, \quad x_2 = \frac{1}{k_1^2} \quad ii) x_1 = \frac{1}{k_1^2}, \quad x_2 = x^2, \quad (6.15)$$

and from Eq.(4.13), Eq.(4.14) and Eq.(4.15), we have two solutions

$$i) x_1 = x^2, \quad x_2 = \frac{1}{k_2^2}, \quad ii) x_1 = \frac{1}{k_2^2}, \quad x_2 = x^2, \quad (6.16)$$

Combining these relations, we have

$$i)x_1 = x^2, \quad x_2 = \frac{1}{k_0^2} = \frac{1}{k_1^2} = \frac{1}{k_2^2} = (\text{const.}), \quad (6.17)$$

$$ii)x_1 = \frac{1}{k_0^2} = \frac{1}{k_1^2} = \frac{1}{k_2^2} = (\text{const.}), \quad x_2 = x^2. \quad (6.18)$$

We consider the case *i*). Then, by using  $x_1 = x^2$  and  $x_2 = 1/k_0^2 = 1/k_1^2 = 1/k_2^2 = \text{const.}$ , the differential equations Eq.(5.25) and Eq.(5.26) reduces to only one differential equation

$$\begin{aligned} du &= \frac{(P + Qx_1)dx_1}{\sqrt{\tilde{f}_5(x_1)}} = \frac{(1 - k_0^2 x_1)dx_1}{2\sqrt{\tilde{f}_5(x_1)}} \\ &= \frac{(1 - k_0^2 x^2)xdx}{\sqrt{x^2(1 - x^2)(1 - k_0^2 x^2)^3}} = \frac{dx}{\sqrt{(1 - x^2)(1 - k_0^2 x^2)}}. \end{aligned} \quad (6.19)$$

where we use  $\tilde{f}_5(x) = x_1(1 - x_1)(1 - k_0^2 x_1)^3$ . Then we have the differential equation of the elliptic function.

## 7 Summary and discussion

The Ising model, which is parameterized by the elliptic function, satisfies the two dimensional integrability equation with difference property. We form the addition formula of the elliptic function into the integrable  $SU(2)$  group structure in the previous paper.

We expect that the addition formula of the Abelian function with any genus will form some integrable Lie group structure. For that purpose, we first study the Jacobi's inversion problem of the hyperelliptic integral with genus two case in this paper, to have the hint to understand the structure of the addition formula of the Abelian function with genus two. Even for the genus one or the genus two case, the essence of the addition formula of the elliptic function comes from the Riemann theta identity, so that we expect some integrable  $SU(2)$  structure for the addition formula of the hyperelliptic theta function with two variables.



## References

- [1] R.J. Baxter, "Exactly Solved Models in Statistical Mechanics", (Academic, New York), 1982.
- [2] L. Onsager, *Phys. Rev.*, **60** (1944), 117.
- [3] K. Shigemoto, *Tezukayama Academic Review*, **No.17** (2011), 15.
- [4] K. Shigemoto, *Tezukayama Academic Review*, **No.19** (2013), 1.
- [5] A. Göpel *J. reine angew. Math.*, **35** (1847), 277-312.
- [6] A. Göpel edited by H. Weber and A. Witting, "Entwurf einer Theorie der Abel'schen Transcendenten earster Ordnung", (Leipzig, W. Engelmann), 1895.
- [7] G. Rosenhain, *Mémoires des savants étrangers*, **XI** (1851), 362-468.
- [8] G. Rosenhain edited by H.Weber and A.Witting "Abhandlung über die Functionen zweiter Variabler mit vier Perioden, welche die Inversen sind der ultra-elliptischen Integrale erster klasse", (Leipzig, W. Engelmann), 1895.
- [9] G. Rosenhain, *J. reine angew. Math.*, **40** (1850), 319-360.
- [10] E. Kossak, "Das Additionstheorem Der Ultra-Elliptischen Functionen Erster Ordnung", (Berlin, Nicolai'sche Verlagsbuchhandlung), 1871.

## A Property of theta function

### A.1 Even odd property

$$\vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (-u, -v) = (-1)^{ab+cd} \vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (u, v) \quad (\text{A.1})$$

Then the following 6 theta functions  $\vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, v)$ ,  $\vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, v)$ ,  $\vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v)$ ,

$\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (u, v)$ ,  $\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (u, v)$ ,  $\vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u, v)$  are odd function, so that we have

$$\vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (0, 0) = 0, \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (0, 0) = 0, \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) = 0, \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (0, 0) = 0,$$

$$\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (u, v) = 0, \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u, v) = 0, \text{ and the rest 10 theta functions are even function.}$$

### A.2 Half periodic property I

$$\vartheta \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} (u + \frac{1}{2}, v) = \vartheta \begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix} (u, v), \quad (\text{A.2})$$

$$\vartheta \begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix} (u + \frac{1}{2}, v) = \vartheta \begin{bmatrix} 1 & c \\ 1 & d \end{bmatrix} (u, v), \quad (\text{A.3})$$

$$\vartheta \begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix} (u + \frac{1}{2}, v) = \vartheta \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} (u, v), \quad (\text{A.4})$$

$$\vartheta \begin{bmatrix} 1 & c \\ 1 & d \end{bmatrix} (u + \frac{1}{2}, v) = (-1) \vartheta \begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix} (u, v). \quad (\text{A.5})$$

### A.3 Half periodic property II

$$\vartheta \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} (u + \frac{\tau_1}{2}, v + \frac{\tau_{12}}{2}) = e^{-i\pi\tau_1/4 - i\pi u} \vartheta \begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix} (u, v), \quad (\text{A.6})$$

$$\vartheta \begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix} (u + \frac{\tau_1}{2}, v + \frac{\tau_{12}}{2}) = e^{-i\pi\tau_1/4 - i\pi u} \vartheta \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} (u, v), \quad (\text{A.7})$$

$$\vartheta \begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix} (u + \frac{\tau_1}{2}, v + \frac{\tau_{12}}{2}) = (-\sqrt{-1})e^{-i\pi\tau_1/4-i\pi u} \vartheta \begin{bmatrix} 1 & c \\ 1 & d \end{bmatrix} (u, v), \quad (\text{A.8})$$

$$\vartheta \begin{bmatrix} 1 & c \\ 1 & d \end{bmatrix} (u + \frac{\tau_1}{2}, v + \frac{\tau_{12}}{2}) = (-1\sqrt{-1})e^{-i\pi\tau_1/4-i\pi u} \vartheta \begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix} (u, v). \quad (\text{A.9})$$

#### A.4 Half periodic property III

$$\vartheta \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} (u + \frac{\tau_1}{2} + \frac{1}{2}, v + \frac{\tau_{12}}{2}) = (-\sqrt{-1})e^{-i\pi\tau_1/4-i\pi u} \vartheta \begin{bmatrix} 1 & c \\ 1 & d \end{bmatrix} (u, v), \quad (\text{A.10})$$

$$\vartheta \begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix} (u + \frac{\tau_1}{2} + \frac{1}{2}, v + \frac{\tau_{12}}{2}) = (-\sqrt{-1})e^{-i\pi\tau_1/4-i\pi u} \vartheta \begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix} (u, v), \quad (\text{A.11})$$

$$\vartheta \begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix} (u + \frac{\tau_1}{2} + \frac{1}{2}, v + \frac{\tau_{12}}{2}) = e^{-i\pi\tau_1/4-i\pi u} \vartheta \begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix} (u, v), \quad (\text{A.12})$$

$$\vartheta \begin{bmatrix} 1 & c \\ 1 & d \end{bmatrix} (u + \frac{\tau_1}{2} + \frac{1}{2}, v + \frac{\tau_{12}}{2}) = (-1)e^{-i\pi\tau_1/4-i\pi u} \vartheta \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} (u, v). \quad (\text{A.13})$$

#### A.5 Periodic property I

$$\vartheta \begin{bmatrix} 0 & c \\ b & d \end{bmatrix} (u + 1, v) = \vartheta \begin{bmatrix} 0 & c \\ b & d \end{bmatrix} (u, v), \quad (\text{A.14})$$

$$\vartheta \begin{bmatrix} 1 & c \\ b & d \end{bmatrix} (u + 1, v) = (-1)\vartheta \begin{bmatrix} 1 & c \\ b & d \end{bmatrix} (u, v). \quad (\text{A.15})$$

#### A.6 Periodic property II

$$\vartheta \begin{bmatrix} a & c \\ 0 & d \end{bmatrix} (u + \tau_1, v + \tau_{12}) = e^{-i\pi\tau_1-2i\pi u} \vartheta \begin{bmatrix} a & c \\ 0 & d \end{bmatrix} (u, v), \quad (\text{A.16})$$

$$\vartheta \begin{bmatrix} a & c \\ 1 & d \end{bmatrix} (u + \tau_1, v + \tau_{12}) = (-1)e^{-i\pi\tau_1-2i\pi u} \vartheta \begin{bmatrix} a & c \\ 1 & d \end{bmatrix} (u, v). \quad (\text{A.17})$$

## B Various theta identity

$$\begin{aligned} \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v) &= \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u, v) \\ &+ \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u, v) + \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u, v), \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v) &= \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u, v) \\ &+ \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u, v) + \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v), \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v) &= \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u, v) \\ &+ \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u, v) - \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, v). \end{aligned} \quad (\text{B.3})$$

### B.1 Derivation of Eq.(B.1) and Eq.(B.2)

Substituting  $u_1 = u_2 = u$ ,  $u_3 = u_4 = 0$ ,  $v_1 = v_2 = v$ ,  $v_3 = v_4 = 0$ , which gives  $\tilde{u}_1 = \tilde{u}_2 = u$ ,  $\tilde{u}_3 = \tilde{u}_4 = 0$ ,  $\tilde{v}_1 = v_2 = v$ ,  $\tilde{v}_3 = \tilde{v}_4 = 0$  in Eq.(3.26), that is,  $2\tilde{M} = M + M' + M'' + M'''$ , we have

$$\begin{aligned} &\left\{ \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v) - \left( \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u, v) \right. \right. \\ &\quad \left. \left. + \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u, v) + \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u, v) \right) \right\} \\ &+ \left\{ \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u, v) - \left( \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (u, v) \right. \right. \\ &\quad \left. \left. + \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u, v) - \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u, v) \right) \right\} = 0. \end{aligned} \quad (\text{B.4})$$

Next, substituting  $u_1 = u_2 = u$ ,  $u_3 = u_4 = 0$ ,  $v_1 = v + 1$ ,  $v_2 = v$ ,  $v_3 = v_4 = 0$ , which gives  $\tilde{u}_1 = \tilde{u}_2 = u$ ,  $\tilde{u}_3 = \tilde{u}_4 = 0$ ,  $\tilde{v}_1 = \tilde{v}_2 = v + 1/2$ ,  $\tilde{v}_3 = \tilde{v}_4 = 1/2$  in Eq.(3.28), that is,  $2\tilde{M}'' = M - M' + M'' - M'''$ , we have

$$\begin{aligned}
& \left\{ \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) - \left( \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u,v) \right. \right. \\
& \quad \left. \left. + \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u,v) + \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u,v) \right) \right\} \\
& - \left\{ \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v) - \left( \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (u,v) \right. \right. \\
& \quad \left. \left. + \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u,v) \right) \right\} = 0. \quad (\text{B.5})
\end{aligned}$$

Combining Eq.(B.4) and Eq.(B.5), we have Eq.(B.1) and Eq.(B.2).

## B.2 Derivation of Eq.(B.3)

Substituting  $u_1 = u + 1$ ,  $u_2 = u$ ,  $u_3 = 1/2$ ,  $u_4 = 1/2$ ,  $v_1 = v_2 = v + 1/2$ ,  $v_3 = 1/2$ ,  $v_4 = -1/2$ , which gives  $\tilde{u}_1 = u + 1$ ,  $\tilde{u}_2 = u$ ,  $\tilde{u}_3 = 1/2$ ,  $\tilde{u}_4 = 1/2$ ,  $\tilde{v}_1 = \tilde{v}_2 = v + 1/2$ ,  $\tilde{v}_3 = 1/2$ ,  $\tilde{v}_4 = -1/2$  in Eq.(3.26), that is,  $2\tilde{M} = M + M' + M'' + M'''$ , we have

$$\begin{aligned}
& \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v) = \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \\
& + \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u,v). \quad (\text{B.6})
\end{aligned}$$

We further replace  $u \rightarrow u$ ,  $v \rightarrow v + 1/2$ , we have

$$\begin{aligned}
& \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) = \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u,v) \\
& + \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,v). \quad (\text{B.7})
\end{aligned}$$

which gives Eq.(B.3).

## C Parameterization of constant

In order that these parameterization is consistent, we must have

$$\frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)} k_0^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} k_1^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)} k_2^2, \quad (\text{C.1})$$

$$\frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0)} \frac{k_0^2}{1 - k_0^2} = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)} \frac{k_1^2}{1 - k_1^2} = \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)} \frac{k_2^2}{1 - k_2^2}, \quad (\text{C.2})$$

which gives

$$k_0^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v)}, \quad \frac{k_0^2}{1 - k_0^2} = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)} \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}, \quad (\text{C.3})$$

$$k_1^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)} \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad \frac{k_1^2}{1 - k_1^2} = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0)} \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}, \quad (\text{C.4})$$

$$k_2^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)} \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad \frac{k_2^2}{1 - k_2^2} = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0)} \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)}. \quad (\text{C.5})$$

Consistency of  $k_0^2$  in Eq.(C.3), consistency of  $k_1^2$  in Eq.(C.4), consistency of  $k_2^2$  in Eq.(C.5), give the following relation

$$1 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)} - \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (\text{C.6})$$

$$1 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)} - \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (\text{C.7})$$

$$1 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} - \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}. \quad (\text{C.8})$$

Eq.(C.6) is derived from Eq.(B.1) by putting  $u = \tau_{12}/2$ ,  $v = \tau_2/2$ . Eq.(C.7) is derived from Eq.(B.3) by putting  $u = \tau_{12}/2$ ,  $v = \tau_2/2$ . Eq.(C.8) is derived from Eq.(B.3) by putting  $u = 0$ ,  $v = 0$ .

As we use the expression  $k_0'^2 = 1 - k_0^2$ ,  $k_1'^2 = 1 - k_1^2$ ,  $k_2'^2 = 1 - k_2^2$ ,  $k_{01}^2 = k_0^2 - k_1^2$ ,  $k_{02}^2 = k_0^2 - k_2^2$ ,  $k_{12}^2 = k_1^2 - k_2^2$ , expressed with the ration of theta function with zero argument.

From Eq.(4.21), Eq.(4.22), Eq.(4.23), we have

$$k_0'^2 = 1 - k_0^2 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (\text{C.9})$$

$$k_1'^2 = 1 - k_1^2 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (\text{C.10})$$

$$k_2'^2 = 1 - k_2^2 = \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}, \quad (\text{C.11})$$

and  $k_{01}^2 = k_0^2 - k_1^2$ ,  $k_{02}^2 = k_0^2 - k_2^2$ ,  $k_{12}^2 = k_1^2 - k_2^2$ , are expressed with the ration of theta function with zero argument.

We have

$$\begin{aligned}
k_{01}^2 = k_0^2 - k_1^2 &= \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)} \\
&\times \frac{\left( \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \right)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)} \\
&= \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}, \tag{C.12}
\end{aligned}$$

by using Eq.(B.1) with putting  $u = \tau_1/2$ ,  $v = \tau_{12}/2 + 1/2$ .

Similarly, we have We have

$$\begin{aligned}
k_{02}^2 = k_0^2 - k_2^2 &= \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)} \\
&\times \frac{\left( \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) - \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \right)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)} \\
&= \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}, \tag{C.13}
\end{aligned}$$



by using Eq.(B.2) with putting  $u = \tau_1/2$ ,  $v = \tau_{12}/2 + 1/2$ .  
Similarly, we have

$$\begin{aligned}
k_{12}^2 &= k_1^2 - k_2^2 = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0)} \\
&\times \frac{\left( \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) - \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \right)}{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0)} \\
&= \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0)}, \tag{C.14}
\end{aligned}$$

by using Eq.(B.1) with putting  $u = \tau_1/2 + \tau_{12}/2$ ,  $v = \tau_2/2 + \tau_{12}/2$ .

## D Parameterization of the ratio of other theta function

### D.1 Derivation of Eq.(4.35)[6-th parameterization]

In this appendix, we will first show

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = -\frac{x_1 x_2 (1 - x_1)(1 - x_2)}{k'_0 k'_1 k'_2 (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{x_1(1 - x_1)} \mp \frac{\sqrt{f_5(x_2)}}{x_2(1 - x_2)} \right\}^2. \tag{D.1}$$

When we take the square root, in order that the right hand side of the above is symmetric for  $x_1 \leftrightarrow x_2$ , we take minus sign of  $\mp$  in the above, and we have

$$\frac{\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u, v)}{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)} = \sqrt{-1} \frac{\sqrt{x_1 x_2 (1-x_1)(1-x_2)}}{\sqrt{k'_0 k'_1 k'_2 (x_2 - x_1)}} \left\{ \frac{\sqrt{f_5(x_1)}}{x_1(1-x_1)} - \frac{\sqrt{f_5(x_2)}}{x_2(1-x_2)} \right\}. \quad (\text{D.2})$$

In that purpose, we want to derive the algebraic equation of  $\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u, v)$  and  $\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)$ . For that purpose, we use the identity

$$\begin{aligned} & \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u, v) \\ & - \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (u, v) \\ & - \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u, v) = 0, \end{aligned} \quad (\text{D.3})$$

where Eq.(D.3) is obtained by replacing  $u_1 \rightarrow u + \tau_{12}/2$ ,  $u_2 \rightarrow u$ ,  $u_3 \rightarrow \tau_{12}/2$ ,  $u_4 \rightarrow 0$ ,  $v_1 \rightarrow v + \tau_2/2 + 1/2$ ,  $v_2 \rightarrow v + 1/2$ ,  $v_3 \rightarrow \tau_2/2$ ,  $v_4 \rightarrow 0$ , which gives  $\tilde{u}_1 \rightarrow u + \tau_{12}/2$ ,  $\tilde{u}_2 \rightarrow u$ ,  $\tilde{u}_3 \rightarrow \tau_{12}/2$ ,  $\tilde{u}_4 \rightarrow 0$ ,  $\tilde{v}_1 \rightarrow v + \tau_2/2 + 1/2$ ,  $\tilde{v}_2 \rightarrow v + 1/2$ ,  $\tilde{v}_3 \rightarrow \tau_2/2$ ,  $\tilde{v}_4 \rightarrow 0$  in Eq.(3.26), that is,  $2\tilde{M} = M + M' + M'' + M'''$ .

In this identity, we use Eq.(4.32) for  $\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (u, v)$ , and Eq.(4.31) for  $\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v)$ , so that it is necessary to use the identity to connect  $\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u, v)$  and  $\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (u, v)$  with  $\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u, v)$ ,  $\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)$ ,  $\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (u, v)$ ,  $\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v)$ .

Then we use the following such identity

$$\begin{aligned} & \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u, v) = \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \\ & + \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) - \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u, v), \quad (\text{D.4}) \\ & \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (u, v) = \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \end{aligned}$$

$$-\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) + \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v), \quad (\text{D.5})$$

where Eq.(D.4) is obtained by replacing  $u \rightarrow u + \tau_1/2$ ,  $v \rightarrow v + \tau_{12}/2 + 1/2$  in Eq.(B.3), and Eq.(D.5) is obtained by replacing  $u \rightarrow u + 1/2$ ,  $v \rightarrow v + 1/2$  in Eq.(B.3).

After the straightforward but quite tedious calculation, we have

$$\begin{aligned} & \frac{\vartheta^4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^4 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + 2 \left\{ \frac{F_{01}(x_2)F_{234}(x_1) + F_{01}(x_1)F_{234}(x_2)}{k'_0 k'_1 k'_2 (x_2 - x_1)^2} \right\} \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\ & + \left\{ \frac{F_{01}(x_2)F_{234}(x_1) - F_{01}(x_1)F_{234}(x_2)}{k'_0 k'_1 k'_2 (x_2 - x_1)^2} \right\}^2 = 0, \end{aligned} \quad (\text{D.6})$$

where we use  $F_{01}(x) = x(1-x)$  and  $F_{234}(x) = (1 - k_0^2 x)(1 - k_1^2 x)(1 - k_2^2 x) = f_5(x)/F_{01}(x)$ . From this we have

$$\frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} = -\frac{x_1 x_2 (1-x_1)(1-x_2)}{k'_0 k'_1 k'_2 (x_2 - x_1)^2} \left\{ \frac{\sqrt{f_5(x_1)}}{x_1(1-x_1)} \mp \frac{\sqrt{f_5(x_2)}}{x_2(1-x_2)} \right\}^2, \quad (\text{D.7})$$

which gives Eq.(4.35).

## D.2 Check of Eq.(4.36)[7-th parameterization]

We sketch how to check Eq.(4.36)  $\sim$  Eq.(4.44) by using Eq.(4.35) and various theta identities. Replacing  $u \rightarrow u + \tau_1/2$  and  $v \rightarrow v + \tau_{12}/2 + 1/2$  in Eq.(B.3), we have

$$\begin{aligned} & \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v) = -\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \\ & + \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v) + \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u,v), \end{aligned} \quad (\text{D.8})$$

which gives

$$\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}$$

$$= \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}. \quad (\text{D.9})$$

By using Eq.(4.30), Eq.(4.31), Eq.(4.35), Eq.(4.36), after the straightforward but tedious calculation, we have

$$\begin{aligned} & (\text{Left} - \text{hand side of Eq.(D.9)}) = (\text{Right} - \text{hand side of Eq.(D.9)}) \\ &= \frac{k_0^2 k_2 k_{12}}{k_1'^2 k_2' k_{02}} \left\{ (k_1^2 k_2^2 - k_1^2 - k_2^2) x_1 x_2 + x_1 + x_2 - 1 \right\}, \end{aligned}$$

which gives Eq.(4.36).

### D.3 Check of Eq.(4.37)[8-th parameterization]

Similarly, by replacing  $u \rightarrow u + \tau_1/2$  and  $v \rightarrow v + \tau_{12}/2 + 1/2$  in Eq.(B.1), we have

$$\begin{aligned} & \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} - \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\ &= - \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} - \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}. \quad (\text{D.10}) \end{aligned}$$

By using Eq.(4.30), Eq.(4.31), Eq.(4.35), Eq.(4.37), after the straightforward but tedious calculation, we have

$$\begin{aligned} & (\text{Left} - \text{hand side of Eq.(D.10)}) = (\text{Right} - \text{hand side of Eq.(D.10)}) \\ &= - \frac{k_0 k_1 k_2}{k_0' k_1' k_2'} \left\{ (k_0^2 k_2^2 - k_0^2 - k_2^2) x_1 x_2 + x_1 + x_2 - 1 \right\}, \end{aligned}$$

which gives Eq.(4.37).

#### D.4 Check of Eq.(4.38)[9-th parameterization]

Similarly, by replacing  $u \rightarrow u + \tau_1/2$  and  $v \rightarrow v + \tau_{12}/2 + 1/2$  in Eq.(B.2), we have

$$\begin{aligned}
& \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\
&= - \frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}. \quad (\text{D.11})
\end{aligned}$$

By using Eq.(4.30), Eq.(4.31), Eq.(4.35), Eq.(4.38), after the straightforward but tedious calculation, we have

$$\begin{aligned}
& (\text{Left} - \text{hand side of Eq.(D.11)}) = (\text{Right} - \text{hand side of Eq.(D.11)}) \\
&= - \frac{k_0 k_2^2 k_{01}}{k_0' k_1'^2 k_{02}} \left\{ (k_0^2 k_1^2 - k_0^2 - k_1^2) x_1 x_2 + x_1 + x_2 - 1 \right\},
\end{aligned}$$

which gives Eq.(4.38).

#### D.5 Check of Eq.(4.39)[10-th parameterization]

Similarly, by replacing  $u \rightarrow u$  and  $v \rightarrow v + 1/2$  in Eq.(B.3), we have

$$\begin{aligned}
& \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\
&= \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}. \quad (\text{D.12})
\end{aligned}$$

By using Eq.(4.31), Eq.(4.35), Eq.(4.39), after the straightforward but tedious calculation, we have

$$\begin{aligned} & (\text{Left} - \text{hand side of Eq.(D.12)}) = (\text{Right} - \text{hand side of Eq.(D.12)}) \\ & = -\frac{k_0 k_{12}}{k_1 k_1'^2 k_2' k_{02}} \left\{ k_1^2 k_2^2 x_1 x_2 - k_1^2 k_2^2 (x_1 + x_2) + k_1^2 + k_2^2 - 1 \right\}, \end{aligned}$$

which gives Eq.(4.39).

## D.6 Check of Eq.(4.40)[11-th parameterization]

Similarly, by replacing  $u \rightarrow u$  and  $v \rightarrow v + 1/2$  in Eq.(B.1), we have

$$\begin{aligned} & -\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\ & = \frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}. \end{aligned} \tag{D.13}$$

By using Eq.(4.31), Eq.(4.35), Eq.(4.40), after the straightforward but tedious calculation, we have

$$\begin{aligned} & (\text{Left} - \text{hand side of Eq.(D.13)}) = (\text{Right} - \text{hand side of Eq.(D.13)}) \\ & = -\frac{1}{k_0' k_1' k_2'} \left\{ k_0^2 k_2^2 x_1 x_2 - k_0^2 k_2^2 (x_1 + x_2) + k_0^2 + k_2^2 - 1 \right\}, \end{aligned}$$

which gives Eq.(4.40).

## D.7 Check of Eq.(4.41)[12-th parameterization]

Similarly, by replacing  $u \rightarrow u$  and  $v \rightarrow v + 1/2$  in Eq.(B.2), we have

$$\begin{aligned}
& -\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + \frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\
& = \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + \frac{\vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}. \tag{D.14}
\end{aligned}$$

By using Eq.(4.31), Eq.(4.35), Eq.(4.41), after the straightforward but tedious calculation, we have

$$\begin{aligned}
& (\text{Left} - \text{hand side of Eq.(D.14)}) = (\text{Right} - \text{hand side of Eq.(D.14)}) \\
& = -\frac{k_2 k_{01}}{k'_0 k_1 k_1'^2 k_{02}} \left\{ k_0^2 k_1^2 x_1 x_2 - k_0^2 k_1^2 (x_1 + x_2) + k_0^2 + k_1^2 - 1 \right\},
\end{aligned}$$

which gives of Eq.(4.41).

## D.8 Check of Eq.(4.42)[13-th parameterization]

Similarly, by replacing  $u \rightarrow u + 1/2$  and  $v \rightarrow v + 1/2$  in Eq.(B.3), we have

$$\begin{aligned}
& -\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\
& = -\frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + \frac{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}. \tag{D.15}
\end{aligned}$$

By using Eq.(4.30), Eq.(4.35), Eq.(4.42), after the straightforward but tedious calculation, we have

$$\begin{aligned} & (\text{Left} - \text{hand side of Eq. (D.15)}) = (\text{Right} - \text{hand side of Eq. (D.15)}) \\ & = -\frac{k_0 k'_0 k_{12}}{k_1 k'_1 k_{02}} \left\{ k_1^2 k_2^2 x_1 x_2 - 1 \right\}, \end{aligned}$$

which gives Eq.(4.42).

## D.9 Check of Eq.(4.43)[14-th parameterization]

Similarly, by replacing  $u \rightarrow u + 1/2$  and  $v \rightarrow v + 1/2$  in Eq.(B.3), we have

$$\begin{aligned} & \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) - \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\ & = -\frac{\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} + 1. \end{aligned} \tag{D.16}$$

By using Eq.(4.30), Eq.(4.35), Eq.(4.43), after the straightforward but tedious calculation, we have

$$\begin{aligned} & (\text{Left} - \text{hand side of Eq. (D.16)}) = (\text{Right} - \text{hand side of Eq. (D.16)}) \\ & = -\left\{ k_0^2 k_2^2 x_1 x_2 - 1 \right\}, \end{aligned}$$

which gives Eq.(4.43).



## D.10 Check of Eq.(4.44)[15-th parameterization]

Similarly, by replacing  $u \rightarrow u + 1/2$  and  $v \rightarrow v + 1/2$  in Eq.(B.2), we have

$$\begin{aligned}
& \frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u,v) + \vartheta^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (u,v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) + \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v)} \\
&= -\frac{\vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) + \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) + \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0)}. \tag{D.17}
\end{aligned}$$

By using Eq.(4.30), Eq.(4.35), Eq.(4.44), after the straightforward but tedious calculation, we have

$$\begin{aligned}
& (\text{Left - hand side of Eq.(D.17)}) = (\text{Right - hand side of Eq.(D.17)}) \\
&= -\frac{k_2 k'_2 k_{01}}{k_1 k'_1 k_{02}} \left\{ k_0^2 k_1^2 x_1 x_2 - 1 \right\},
\end{aligned}$$

which gives Eq.(4.44).

## E Addition formula of theta function and differential formula

### E.1 Addition theorem of theta function

We will show the following addition theorem

$$\begin{aligned}
& 1) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \left\{ \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u+u', v+v') \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u-u', v-v') \right. \\
& \left. - \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u+u', v+v') \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u-u', v-v') \right\} \\
&= 2 \left\{ \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (u',v') \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u',v') \right.
\end{aligned}$$

$$-\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u', v') \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (u', v') \Big\}, \quad (\text{E.1})$$

$$\begin{aligned} & 2) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \left\{ \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u + u', v + v') \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u - u', v - v') \right. \\ & \left. - \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u + u', v + v') \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u - u', v - v') \right\} \\ & = 2 \left\{ -\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u', v') \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u', v') \right. \\ & \left. + \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u', v') \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u', v') \right\}. \quad (\text{E.2}) \end{aligned}$$

Replacing  $u_1 \rightarrow u + \tau_1/2 + 1/2$ ,  $v_1 \rightarrow v + \tau_{12}/2 + 1$ ,  $u_2 \rightarrow u + 1/2$ ,  $v_2 \rightarrow v$ ,  $u_3 \rightarrow u' + \tau_1/2$ ,  $v_3 \rightarrow v' + \tau_{12}/2$ ,  $u_4 \rightarrow u'$ ,  $v_4 \rightarrow v'$ , which gives  $\tilde{u}_1 \rightarrow u + u' + \tau_1/2 + 1/2$ ,  $\tilde{v}_1 \rightarrow v + v' + \tau_{12}/2 + 1/2$ ,  $\tilde{u}_2 \rightarrow u - u' + 1/2$ ,  $\tilde{v}_2 \rightarrow v - v' + 1/2$ ,  $\tilde{u}_3 \rightarrow \tau_1/2$ ,  $v_3 \rightarrow \tau_{12}/2 + 1/2$ ,  $u_4 \rightarrow 0$ ,  $v_4 \rightarrow 1/2$ , in Eq.(3.25), that is,  $2M''' = \tilde{M} - \tilde{M}' - \tilde{M}'' + \tilde{M}'''$ . In this case, we have  $\tilde{M}'' = 0$ ,  $\tilde{M}''' = 0$ , so that we have  $2M''' = \tilde{M} - \tilde{M}'$ . After using the transformation property, we have Eq.(E.1). Similarly, replacing  $u_1 \rightarrow u$ ,  $v_1 \rightarrow v + 1$ ,  $u_2 \rightarrow u + \tau_1/2 + 1/2$ ,  $v_2 \rightarrow v + \tau_{12}/2$ ,  $u_3 \rightarrow u'$ ,  $v_3 \rightarrow v'$ ,  $u_4 \rightarrow u' + \tau_1/2 - 1/2$ ,  $v_4 \rightarrow v' + \tau_{12}/2$ , which gives  $\tilde{u}_1 \rightarrow u + u' + \tau_1/2$ ,  $\tilde{v}_1 \rightarrow v + v' + \tau_{12}/2 + 1/2$ ,  $\tilde{u}_2 \rightarrow u - u' + 1/2$ ,  $\tilde{v}_2 \rightarrow v - v' + 1/2$ ,  $\tilde{u}_3 \rightarrow -\tau_1/2$ ,  $v_3 \rightarrow -\tau_{12}/2 + 1/2$ ,  $u_4 \rightarrow -1/2$ ,  $v_4 \rightarrow 1/2$ , in Eq.(3.22), that is,  $2M''' = \tilde{M} + \tilde{M}' + \tilde{M}'' + \tilde{M}'''$ . In this case, we have  $\tilde{M}' = 0$ ,  $\tilde{M}''' = 0$ , so that we have  $2M = \tilde{M} + \tilde{M}''$ . After using the transformation property, we have Eq.(E.2).

## E.2 Derivative formula I

We calculate

$$\begin{aligned} & \frac{\partial}{\partial u} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right) \\ & = \frac{\partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) - \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \partial_u \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)}, \quad (\text{E.3}) \end{aligned}$$

by using the addition theorem.

For this purpose, we use the following addition formula Eq.(E.1) and put  $u' = du$ ,  $v' = 0$ ,

$$\begin{aligned}
& \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \left\{ \left( \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) + \frac{\partial}{\partial u} \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) du \right) \right. \\
& \times \left( \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) - \frac{\partial}{\partial u} \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) du \right) - \left( \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) + \frac{\partial}{\partial u} \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) du \right) \\
& \times \left. \left( \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) - \frac{\partial}{\partial u} \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) du \right) \right\} \\
& = 2 \left\{ \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,0)|_0 du \right. \\
& \left. - \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v) \partial_u \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,0)|_0 du \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \right\}, \tag{E.4}
\end{aligned}$$

which gives the expression

$$\begin{aligned}
& \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \frac{\partial}{\partial u} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \right) \\
& = \left( \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,0)_0 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \right. \\
& \left. - \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \partial_u \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,0)_0 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v) \right), \tag{E.5}
\end{aligned}$$

which is used in Eq.(5.9) where we denote  $\partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,0)|_0$  for  $\frac{\partial}{\partial u} \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,0)|_{u=0}$  etc. Similarly, by putting  $u' = 0$ ,  $v' = dv$ , we have the expression

$$\begin{aligned}
& \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \frac{\partial}{\partial v} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u,v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u,v) \right) \\
& = \left( \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \partial_v \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (0,v)_0 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \right. \\
& \left. - \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0,0) \partial_v \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (0,v)_0 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v) \right). \tag{E.6}
\end{aligned}$$

### E.3 Derivative formula II

We calculate

$$\begin{aligned} & \frac{\partial}{\partial u} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right) \\ &= \frac{\partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) - \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) \partial_u \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)}{\vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v)}, \end{aligned} \quad (\text{E.7})$$

by using the addition theorem.

For this purpose, we use the following addition formula Eq.(E.2) and put  $u' = du$ ,  $v' = 0$ ,

$$\begin{aligned} & \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \left\{ \left( \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) + \frac{\partial}{\partial u} \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) du \right) \right. \\ & \times \left( \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) - \frac{\partial}{\partial u} \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) du \right) - \left( \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) + \frac{\partial}{\partial u} \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) du \right) \\ & \times \left. \left( \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) - \frac{\partial}{\partial u} \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) du \right) \right\} \\ &= 2 \left\{ -\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, 0) \Big|_0 du \right. \\ & \left. + \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \partial_u \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, 0) \Big|_0 du \right\}, \end{aligned} \quad (\text{E.8})$$

which gives the expression

$$\begin{aligned} & \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \frac{\partial}{\partial u} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right) \\ &= \left( -\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) \partial_u \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, 0)_0 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u, v) \right. \\ & \left. + \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) \partial_u \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, 0)_0 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u, v) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u, v) \right), \end{aligned} \quad (\text{E.9})$$

which is used in Eq.(5.10).

Similarly, by putting  $u' = 0$ ,  $v' = dv$ , we have the expression

$$\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \frac{\partial}{\partial v} \left( \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u, v) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u, v) \right)$$

$$\begin{aligned}
&= \left( -\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,0) \partial_v \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (0,v)_0 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (u,v) \right. \\
&\quad \left. + \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \partial_v \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (0,v)_0 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u,v) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (u,v) \right). \tag{E.10}
\end{aligned}$$